Taking the explicit hint, we start by showing that f is continuous everywhere. This really comes in at the end, but let's get it out of the way. To do this, we'll need that $f(x-y) = f(x) - f(y)$ for all $x, y \in \mathbb{R}$. Note

$$f(0) = f(0+0) = f(0) + f(0) = 2f(0)$$

and therefore $f(0) = 0$. From there, we get

$$0 = f(x-x) = f(x+(-x)) = f(x) + f(-x)$$

for any x, and therefore $f(x-y) = f(x) - f(y)$ for all $x, y \in \mathbb{R}$.

Suppose that f is continuous at $p \in \mathbb{R}$ and let $\varepsilon > 0$. Choose δ_p so that $|x-p| < \delta_p$ implies $|f(x) - f(p)| < \varepsilon$. For any $x, y \in \mathbb{R}$, we have

$$\begin{aligned} |f(x) - f(y)| &= |f(x-y) + f(p) - f(p)| \\ &= |f(p+(x-y)) - f(p)| \end{aligned}$$

and therefore $|f(x) - f(y)| < \varepsilon$ when

$|x-y| < \delta_p$, since

$$|p+(x-y)-p| = |x-y|.$$

Next, we show that $f(x) = f(1)x$ for all $x \in \mathbb{Q}$. ~~This technically requires two induction proofs.~~ We need $f(1/n) = f(1)/n$ and $f(nx) = nf(x)$ for $n \in \mathbb{N}$. This is obvious and gives $f(x) = f(1)x$ for all rational x.

Finally, suppose x is any real number. There is a sequence $\{x_n\}$ of rationals converging to x. Therefore

$$\begin{aligned}
f(x) &= f\left(\lim_{n\to\infty} x_n\right) \\
&= \lim_{n\to\infty} f(x_n) \\
&= \lim_{n\to\infty} f(1)\,x_n \\
&= \left(\lim_{n\to\infty} x_n\right) f(1) \\
&= f(1)\,x
\end{aligned}$$